Deep inelastic structure functions from supergravity at small x

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ABSTRACT: Deep inelastic structure functions can be calculated from supergravity when the Bjorken parameter x satisfies $x > 1/\sqrt{gN}$. We consider a gauge theory with very large 't Hooft coupling gN in order to investigate the region $x \ll 1$. In this case the center of mass energy is large enough to increase the number of hadronic constituents of the final state. We calculate the structure functions in terms of the number of final hadronic constituents. At small x we find a scaling law similar to geometric scaling but with $\gamma_s = 0.5$ and $\lambda = 1$.

Contents

1. Introduction

Deep inelastic lepton-hadron scattering (DIS) is an important experimental tool for investigating the structure of hadrons. The cross sections for this process can be represented in terms of structure functions depending on the kinematical scalar variables q^2 and x where q^2 is the virtuality of the photon emitted by the charged lepton and x is the Bjorken variable. For large x the structure functions are approximately independent of q^2. This is known as Bjorken scaling. For scattering at small x ($x < 0.01$) the observed total cross section depends on x and q^2 through the variable:

$$\mathcal{T} = q^2 x^\lambda / q_0^2 x_0^\lambda \tag{1.1}$$

where $q_0 = 1 GeV$ and $x_0 = 3 \times 10^{-4}$ and $0.3 < \lambda < 0.4$ [1]. This is called geometric scaling. A study of this phenomena from the point of view of perturbative QCD can be found, for instance, in [2−4].

A description of DIS at large coupling from the point of view of gauge/string duality was formulated by Polchinski and Strassler in [5]. They calculated the DIS structure functions considering a 10-d $AdS_5 \times W$ space with a hard infrared cut off that breaks conformal invariance. Three different kinematical regimes were considered:

- $x > 1/\sqrt{gN}$

- $\exp(-\sqrt{gN}) \ll x \ll (gN)^{-1/2}$

- $x < \exp(-\sqrt{gN})$.

In the first regime the supergravity approximation is valid while in the other two regimes massive string states contribute.

The introduction of a hard cut off in AdS space leads to a discrete mass spectrum for hadrons [6−8] with asymptotically quadratic Regge trajectories. An alternative approach to introduce an infrared cut off, that leads to linear Regge trajectories for the hadronic

spectrum is the soft wall model [9, 10]. The DIS structure functions were calculated recently using this model [11]. Hadronic form factors were also calculated using gauge string duality in for example [12 – 14].

The DIS structure functions obtained in ref. [5] (and also in [11]), from supergravity, were calculated considering that the final hadronic operator has a conformal dimension Δ' equal to the conformal dimension of the initial hadronic operator Δ. As seen in [15], the conformal dimension of a hadronic operator can be identified with the minimum number of constituents of the hadronic state. So the case $\Delta' = \Delta$ represents scattering processes where the final state has the same number of constituents of the initial state.

The initial state in a DIS process contains one hadron. When the center of mass energy $\sqrt{s}$ is larger than the hadronic mass scale we expect more hadrons in the final state. Since $s \sim q^2/x$, this corresponds to small x. The increase in the number of hadrons should be associated with an increase in the number of constituents of the final state.

In the standard AdS/CFT dictionary one associates a bulk supergravity field for each boundary state. So we should introduce one additional bulk field for each new hadron in the final state. We will follow here a simpler phenomenological approach that consists in using just one supergravity field for the final state. The boundary operator dual to this field will have a conformal dimension $\Delta' > \Delta$, representing a final state with more constituents than the initial state. We conjecture that this state evolves to multi-hadronic states.

We calculate within the supergravity approximation the contributions to the DIS structure functions from final states with $\Delta' > \Delta$. We show that these contributions are relevant in the small x regime where they change the structure function dependence on the variables q^2 and x with respect to the results obtained in refs. [5, 11] where $\Delta' = \Delta$. We find a scaling law for the total cross section similar to geometric scaling.

2. Structure functions from supergravity

In deep inelastic scattering a lepton produces a virtual photon of momentum q^μ which interacts with a hadron of momentum P^μ (see figure 1). The experiment detects the final lepton, determining the momentum transfer q^μ, but not the final hadronic state X, just its four-momentum P_X^μ. We can parameterize the process using as dynamical variables the virtuality q^2 and the Bjorken parameter $x \equiv -q^2/2P \cdot q$.

The deep inelastic hadronic tensor for unpolarized scattering is defined as

$$W^{\mu\nu} = i \int d^4y e^{iq\cdot y} \langle P, \mathcal{Q}| \Big[J^\mu(y), J^\nu(0) \Big] |P, \mathcal{Q}\rangle \,, \tag{2.1}$$

where $J^\mu(y)$ is the electromagnetic hadron current and $\mathcal{Q}$ is the electric charge of the initial hadron. This tensor can be decomposed into the structure functions $F_1(x, q^2)$ and $F_2(x, q^2)$ as [16]

$$W^{\mu\nu} = F_1(x, q^2)\Big(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\Big) + \frac{2x}{q^2} F_2(x, q^2)\Big(P^\mu + \frac{q^\mu}{2x}\Big)\Big(P^\nu + \frac{q^\nu}{2x}\Big) \,, \tag{2.2}$$

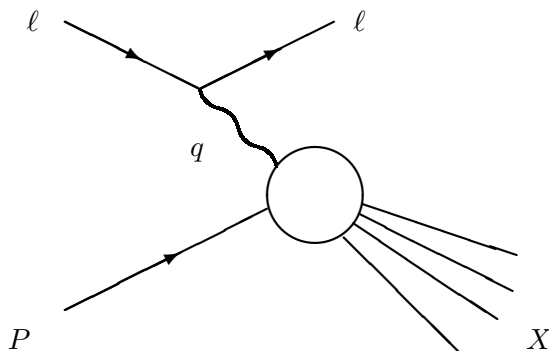where $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$.

Figure 1: Illustrative diagram for a deep inelastic scattering. A lepton ℓ exchanges a virtual photon with a hadron of momentum P.

The DIS cross section is related to the forward Compton scattering tensor

$$T^{\mu\nu} = i \int d^4 y e^{iq\cdot y} \langle P, \mathcal{Q} | T\Big(J^\mu(y)J^\nu(0)\Big) | P, \mathcal{Q} \rangle \,, \qquad (2.3)$$

which can be decomposed as

$$T^{\mu\nu} = \tilde{F}_1(x,q^2)\Big(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\Big) + \frac{2x}{q^2}\tilde{F}_2(x,q^2)\Big(P^\mu + \frac{q^\mu}{2x}\Big)\Big(P^\nu + \frac{q^\nu}{2x}\Big) \,, \qquad (2.4)$$

where $\tilde{F}_1(x,q^2)$ and $\tilde{F}_2(x,q^2)$ are the associated structure functions.

The optical theorem relates the tensors $W^{\mu\nu}$ and $T^{\mu\nu}$ and implies

$$F_{1,2}(x,q^2) \equiv 2\pi \,\mathrm{Im}\, \tilde{F}_{1,2}(x,q^2) \,. \qquad (2.5)$$

The imaginary part of $T^{\mu\nu}$ can be expressed in terms of a sum over the intermediate states X formed in the hadron-photon collision

$$\mathrm{Im}T^{\mu\nu} = 2\pi^2 \sum_X \delta\Big((P+q)^2 - P_X^2\Big)\langle P, \mathcal{Q}|J^\nu(0)|P+q, X\rangle\langle P+q, X|J^\mu(0)|P, \mathcal{Q}\rangle \,. \qquad (2.6)$$

Polchinski and Strassler [5] found prescriptions for calculating this quantity from gauge string duality. When the Bjorken variable satisfies the condition $x > (gN)^{-1/2}$ one can use a supergravity approximation for string theory, since only massless string states contribute. In this work we will be restricted to this regime. The prescription for the case of the scattering of a scalar particle by a virtual photon of polarization η_μ takes the form

$$(2\pi)^4\delta^4(P_X - P - q)\eta_\mu\langle P+q, X|J^\mu(0)|P, \mathcal{Q}\rangle = \int d^{10}x\sqrt{-g}A^m j_m \,, \qquad (2.7)$$

where

$$j_m = i\mathcal{Q}\Big(\Phi_i\partial_m\Phi_X^* - \Phi_X^*\partial_m\Phi_i\Big) \qquad (2.8)$$

is a five dimensional scalar current and $A_m = (A_z, A_\mu)$ is a Kaluza-Klein gauge field. Φ_i and Φ_X are dilaton fields representing the initial and final scalar states. All these fields live in the 10-d space $AdS_5 \times W$, with metric g_{MN}:

$$ds^2 \equiv g_{MN}dx^M dx^N = \frac{R^2}{z^2}(dz^2 + \eta_{\mu\nu}dy^\mu dy^\nu) + R^2 ds_W^2 \,, \qquad (2.9)$$

where W is a compact space. The hard wall condition consists of restricting the radial coordinate z to the region: $0 < z < 1/\Lambda$, where Λ is the minimum hadronic mass in the four dimensional gauge theory.

The equations of motion for the gauge field in the AdS_5 space are

$$\Box A_z - \partial_z \partial_\mu \left(\eta^{\mu\nu} A_\nu \right) = 0 \quad (2.10)$$

$$z\partial_z \left(\frac{1}{z} \partial_z (\eta^{\mu\nu} A_\nu) \right) + \Box(\eta^{\mu\nu} A_\nu) - \eta^{\mu\nu} \partial_\nu \left(z\partial_z \left(\frac{1}{z} A_z \right) + \partial_\alpha (\eta^{\alpha\beta} A_\beta) \right) = 0 \quad (2.11)$$

In order to represent a photon with momentum q^μ and polarization η^μ on the boundary one imposes

$$A_\mu(z = 0, y) = \eta_\mu e^{iq \cdot y} \,. \quad (2.12)$$

The polarization can be decomposed into longitudinal and transversal components: $\eta_\mu = \overset{L}{\eta}_\mu + \overset{T}{\eta}_\mu$ with

$$\overset{L}{\eta}_\mu = \frac{(\eta \cdot q)}{q^2} q_\mu \quad , \quad \overset{T}{\eta}_\mu = \eta_\mu - \frac{(\eta \cdot q)}{q^2} q_\mu \,. \quad (2.13)$$

A natural ansatz for solving the equations of motion with this boundary condition is

$$A_\mu(z, y) = \left(\overset{L}{\eta}_\mu f_L(z) + \overset{T}{\eta}_\mu f_T(z) \right) e^{iq \cdot y} \,, \quad (2.14)$$

where $f_L(z)$ and $f_T(z)$ are functions satisfying the boundary condition $f_T(0) = f_L(0) = 1$.

The equations of motion (2.10) and (2.11) show us that A_z has the same plane wave dependence as A_μ. The first equation gives

$$A_z = -\frac{i}{q^2}(q \cdot \eta^L) e^{iq \cdot y} \partial_z f_L(z) = -\frac{i}{q^2} q^\mu \partial_z A_\mu \,. \quad (2.15)$$

Inserting this solution in eq. (2.11) we find

$$\overset{T}{\eta}_\mu \left[z\partial_z \left(\frac{1}{z} \partial_z f_T(z) \right) - q^2 f_T(z) \right] = 0 \quad (2.16)$$

with solution $f_T(z) = qzK_1(qz)$, where K_1 is a modified Bessel function.

There is no equation for $f_L(z)$. So, the longitudinal part of the gauge field A_μ and the radial component A_z remain undetermined. These components can be determined by imposing a gauge condition. Polchinski and Strassler used the convenient gauge condition

$$\eta^{\mu\nu} \partial_\mu A_\nu + z\partial_z \left(\frac{1}{z} A_z \right) = 0 \,, \quad (2.17)$$

that leads to $f_T(z) = f_L(z)$ and implies the solutions

$$A_\mu = \eta_\mu e^{iq \cdot y} qzK_1(qz)$$
$$A_z = -iq \cdot \eta \, e^{iq \cdot y} zK_0(qz) \,. \quad (2.18)$$

Another possible gauge choice would be $\partial_\mu A^\mu = 0$ that corresponds to a transverse photon ($\overset{L}{\eta}_\mu = 0$) and implies $A_z = 0$.

For the dilaton fields, the AdS_5 part of the equation of motion reads

$$\frac{1}{\sqrt{-g}}\partial_m\left(\sqrt{-g}\partial^m\Phi\right) - m_5^2\Phi = 0. \qquad (2.19)$$

where the five dimensional mass is related to the conformal dimension Δ of the boundary operator dual to Φ by

$$m_5^2 = \frac{\Delta(\Delta - 4)}{R^2}. \qquad (2.20)$$

The solutions for Dirichlet boundary conditions at $z = 1/\Lambda$ for the initial and final scalar states with four dimensional momenta P^μ and P_X^μ are given by

$$\Phi_i = e^{iP\cdot y}\frac{C_i}{R^4}\Lambda z^2 J_{\Delta-2}(\Lambda z)Y(\Omega)$$

$$\Phi_X = e^{iP_X\cdot y}\frac{C_X}{R^4}\Lambda^{1/2}s^{1/4}z^2 J_{\Delta'-2}(s^{1/2}z)Y(\Omega), \qquad (2.21)$$

where $s = -P_X^2, P^2 = -\Lambda^2$, C_i and C_X are normalization constants and $Y(\Omega)$ is the angular part of the solution, that is an eigenstate of the Laplacian in the space W.

The quantities Δ and Δ' are the conformal dimensions of the operators that create the initial and final boundary states. These dimensions can be identified with the number of constituents of the states [15]. Polchinski and Strassler considered the case $\Delta' = \Delta$ that can be interpreted as a DIS process where the number of constituents is preserved.

The condition $\Delta' = \Delta$ implies the conservation of the five dimensional current $j_m = (j_z, j_\mu)$ defined in eq. (2.8):

$$\frac{1}{\sqrt{-g}}\partial_m\left(\sqrt{-g}j^m\right) = 0. \qquad (2.22)$$

Note that the four dimensional part of this current is not conserved: $\partial_\mu j^\mu \neq 0$. Nevertheless, the supergravity interaction between A^m and j_m is equivalent to an interaction between the 4-d A_μ and an effective 4-d conserved current:

$$\int d^{10}x\sqrt{-g}A^m j_m = \int d^{10}x\sqrt{-g}A^\mu\left(j_\mu - \frac{i}{q^2}q_\mu\eta^{\nu\alpha}\partial_\nu j_\alpha\right) \equiv \int d^{10}x\sqrt{-g}A^\mu j_\mu^{\text{eff}}, \qquad (2.23)$$

with $\partial_\mu j_{\text{eff}}^\mu = 0$ (using $q_\mu = (P_X - P)_\mu$). This conservation leads to a $T^{\mu\nu}$ that satisfies the transversality conditions $q_\mu T^{\mu\nu} = 0 = q_\nu T^{\mu\nu}$ as in eq. (2.4).

The structure functions obtained in ref. [5] are

$$F_1^{\Delta'=\Delta}(x,q^2) = 0 \;\; ; \;\; F_2^{\Delta'=\Delta}(x,q^2) = \pi C_0 \mathcal{Q}^2\left(\frac{\Lambda^2}{q^2}\right)^{\Delta-1}x^{\Delta+1}(1-x)^{\Delta-2}. \qquad (2.24)$$

where $C_0 = 2^{2\Delta}\pi|C_i|^2|C_X|^2\Gamma^2(\Delta)$.

It is interesting to note that the above structure function presents Bjorken scaling for $\Delta = 1$, which means that, in this case, the hadron would have only one constituent. Experimentally this behavior is observed for $x > 0.1$.

On the other side, the above structure function at small x have the following behavior

$$\frac{F_2^{\Delta'=\Delta}(x,q^2)}{q^2} \sim \left(q^2 x^\lambda\right)^{\gamma_s-1}, \qquad (2.25)$$

with $\lambda = -\frac{\Delta+1}{\Delta}$ and $\gamma_s = 1 - \Delta$.

3. The $\Delta' > \Delta$ case

In a deep inelastic experiment at small x the center of mass energy squared $s \approx q^2/x$ is large enough to produce multi-hadronic final states. A complete description of such processes should involve introducing more bulk fields to represent these final states. We will not attempt to do this here. We will consider a "final state" still represented by just one dilaton field Φ_X, but with more constituents than the initial state. That means: $\Delta' > \Delta$. Eventually this state will evolve to produce multiple hadrons.

When $\Delta' > \Delta$ the 5-d current defined by eq (2.8) is not conserved:

$$\frac{1}{\sqrt{-g}}\partial_m\left(\sqrt{-g}j^m\right) = \frac{i\mathcal{Q}}{R^2}\Phi_i\Phi_X^*[\Delta'(\Delta'-4) - \Delta(\Delta-4)] \neq 0. \tag{3.1}$$

If we use this non-conserved current to define an interaction of the type $A^m j_m$ as in the previous section, we would find an effective four dimensional current j_μ^{eff} that is non-conserved too. So, we would end up with a non transverse $T^{\mu\nu}$.[1]

We will follow a phenomenological approach and define a modified five dimensional hadronic current:

$$\tilde{j}_m \equiv j_m - v_m\frac{1}{\sqrt{-g}}\partial_n\left(\sqrt{-g}j^n\right). \tag{3.2}$$

A possible choice of v_m that implies a conservation of $\tilde{j}_m$ is

$$v_z = 0 \quad ; \quad v_\mu = i\frac{(P_X - P)_\mu}{(P_X - P)^2}\frac{R^2}{z^2}. \tag{3.3}$$

This current leads to a transverse $T^{\mu\nu}$. Our prescription for the four dimensional current matrix element is

$$(2\pi)^4\delta^4(P_X - P - q)\eta_\mu\langle P+q, X|J^\mu(0)|P,\mathcal{Q}\rangle = \int d^{10}x\sqrt{-g}A^m\tilde{j}_m. \tag{3.4}$$

This interaction term reduces, as in the previous section, to an interaction with an effective four dimensional conserved current

$$\int d^{10}x\sqrt{-g}A^m\tilde{j}_m = \int d^{10}x\sqrt{-g}A^\mu j_\mu^{\text{eff}}$$

$$= (2\pi)^4\delta^4(P_X - P - q)2\mathcal{Q}C_iC_X s^{1/4}\Lambda^{\Delta-1/2}q\eta_\nu\left(p^\mu + \frac{q^\mu}{2x}\right)\mathcal{I}, \tag{3.5}$$

where j_μ^{eff} is again defined by eq. (2.23) and

$$\mathcal{I} \approx \int_0^{1/\Lambda} dz z^\Delta J_{\Delta'-2}(s^{1/2}z)K_1(qz) \approx \int_0^\infty dz z^\Delta J_{\Delta'-2}(s^{1/2}z)K_1(qz). \tag{3.6}$$

The result for this integral is

$$\mathcal{I} = 2^{\Delta-1}\frac{\Gamma(\frac{\Delta'+\Delta}{2})\Gamma(\frac{\Delta'+\Delta}{2}-1)}{\Gamma(\Delta'-1)}\left(\frac{s}{q^2}\right)^{\Delta'/2-1}(q^2)^{-(\Delta+1)/2}$$

$$\times F\left(\frac{\Delta'+\Delta}{2}, \frac{\Delta'+\Delta}{2}-1; \Delta'-1; -\frac{s}{q^2}\right) \tag{3.7}$$

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[1]In spite of this non transversality, it would be possible to extract the structure function F_2 from $T^{\mu\nu}$ by choosing a transverse photon. This would correspond to choosing the gauge condition $\partial_\mu A^\mu = 0$.

where $F(a, b; c; \omega)$ is the Gauss hypergeometric function defined by

$$F(a, b; c; \omega) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \tag{3.8}$$

with $(a)_n = \Gamma(a+n)/\Gamma(a)$.

Using these results in eq. (2.6) we find the contributions to the structure functions coming from final states with dimension Δ':

$$F_1^{\Delta'}(x, q^2) = 0$$
$$F_2^{\Delta'}(x, q^2) = \pi^2 2^{2\Delta} |C_i|^2 |C_X|^2 \mathcal{Q}^2 \left(\frac{\Lambda^2}{q^2}\right)^{\Delta-1} x^{1-\Delta'} (1-x)^{\Delta'-2} \left[\frac{\Gamma(\frac{\Delta'+\Delta}{2})\Gamma(\frac{\Delta'+\Delta}{2}-1)}{\Gamma(\Delta'-1)}\right]^2$$
$$\times \left[F\left(\frac{\Delta'+\Delta}{2}, \frac{\Delta'+\Delta}{2} - 1; \Delta' - 1; -\frac{1-x}{x}\right)\right]^2, \tag{3.9}$$

where we have used the hard wall result for the sum of delta functions

$$\sum_X \delta\left((P+q)^2 - P_X^2\right) = \frac{1}{2\pi s^{1/2}\Lambda}. \tag{3.10}$$

Note that the structure functions will involve the sum over all possible values of Δ'

$$F_2(x, q^2) = F_2^{\Delta'=\Delta}(x, q^2) + \sum_{\Delta'>\Delta} F_2^{\Delta'}(x, q^2). \tag{3.11}$$

Now we introduce the variable $\rho \equiv (\Delta' - \Delta)/2$ to represent the number of extra hadrons (mesons) that can be produced by the "final state". The minimum number of constituents of a hadron is 2. So, we expect non-negative integer numbers for ρ. For example, for $\rho = 0$ the final state can not produce extra hadrons, while for $\rho = 1$ we can expect the production of an extra meson by the "final state". The structure function expressed in terms of ρ is

$$F_2^{\rho}(x, q^2) = \pi^2 2^{2\Delta} |C_i|^2 |C_X|^2 \mathcal{Q}^2 \left(\frac{\Lambda^2}{q^2}\right)^{\Delta-1} x^{1-\Delta}(1-x)^{\Delta-2} \left(\frac{1-x}{x}\right)^{2\rho} \left[\frac{\Gamma(\Delta+\rho)\Gamma(\Delta+\rho-1)}{\Gamma(\Delta+2\rho-1)}\right]^2$$
$$\times \left[F\left(\Delta+\rho, \Delta+\rho-1; \Delta+2\rho-1; -\frac{1-x}{x}\right)\right]^2. \tag{3.12}$$

Using the property

$$F(a, b; c; \omega) = (1-\omega)^{-a} F(a, c-b; c; \frac{\omega}{\omega-1}), \tag{3.13}$$

we have

$$F_2^{\rho}(x, q^2) = F_2^{(\rho=0)}(x, q^2) \left[\frac{\Gamma(\Delta+\rho)\Gamma(\Delta+\rho-1)}{\Gamma(\Delta)\Gamma(\Delta+2\rho-1)}\right]^2 (1-x)^{2\rho} [F(\Delta+\rho, \rho; \Delta+2\rho-1; 1-x)]^2, \tag{3.14}$$

where $F_2^{(\rho=0)}(x, q^2)$ is the structure function of eq. (2.24) obtained in [5].

The $\rho = 1$ case (one extra meson) is simple to calculate because the first and third arguments of the hypergeometric function are equal and simplify in the hypergeometric series of eq. (3.8). We find

$$F_2^{(\rho=1)}(x,q^2) = F_2^{(\rho=0)}(x,q^2)x^{-2}(1-x)^2. \qquad (3.15)$$

The $\rho = 2$ calculation is more subtle. From (3.14) we have that

$$\begin{aligned}
F_2^{(\rho=2)}(x,q^2) &= F_2^{(\rho=0)}(x,q^2)\left(\frac{\Delta}{\Delta+2}\right)^2 (1-x)^4 \left[\sum_{n=0}^{\infty} \frac{(\Delta+2)_n(2)_n}{(\Delta+3)_n}\frac{(1-x)^n}{n!}\right]^2 \\
&= F_2^{(\rho=0)}(x,q^2)\Delta^2(1-x)^4 \left[\sum_{n=0}^{\infty}\frac{(2)_n}{n!}\frac{y^n}{(\Delta+2+n)}\right]^2 \qquad (3.16)
\end{aligned}$$

with $y \equiv 1-x$. The above series can be re-expressed as the integral:

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(2)_n}{n!}\frac{y^n}{(\Delta+2+n)} &= \frac{1}{y^{\Delta+2}}\sum_{n=0}^{\infty}\frac{(2)_n}{n!}\int_0^y dy'(y')^{\Delta+1+n} \\
&= \frac{1}{y^{\Delta+2}}\int_0^y dy'(y')^{\Delta+1}\left[\sum_{n=0}^{\infty}\frac{(2)_n}{n!}(y')^n\right] \\
&= \frac{1}{y^{\Delta+2}}\int_0^y dy'(y')^{\Delta+1}(1-y')^{-2}. \qquad (3.17)
\end{aligned}$$

Defining now $x' \equiv 1-y'$ and substituting (3.17) in (3.16) we obtain

$$\begin{aligned}
F_2^{(\rho=2)}(x,q^2) &= F_2^{(\rho=0)}(x,q^2)\Delta^2(1-x)^{-2\Delta}[\int_x^1 dx'(x')^{-2}(1-x')^{\Delta+1}]^2 \\
&= F_2^{(\rho=0)}(x,q^2)\Delta^2(1-x)^{-2\Delta}\mathcal{S}^2 \qquad (3.18)
\end{aligned}$$

where

$$\mathcal{S} \equiv -1 + \frac{1}{x} + (\Delta+1)\ln x + \sum_{n=2}^{\Delta+1}\binom{\Delta+1}{n}\frac{(-1)^n}{(n-1)}[1-x^{n+1}], \qquad (3.19)$$

and

$$\binom{\Delta+1}{n} = \frac{(\Delta+1)!}{n!(\Delta+1-n)!}. \qquad (3.20)$$

The cases $\rho \geq 3$ can be calculated in a similar way. So it is possible to calculate the contributions for final states corresponding to different numbers of extra hadrons. It is interesting to observe what happens when $x \to 1$ in (3.15) and (3.18). In this limit all the contributions to F_2 vanish, including the case $\rho = 0$. Defining $\epsilon = 1-x \to 0$ it is not difficult to show that the structure functions contributions from $\rho = 1,2$ involve powers of ϵ that are higher than the $\rho = 0$ contribution. So they are negligible near this limit and the structure function can be approximated by considering just $\Delta' = \Delta$. This is consistent with the fact that the case $x = 1$ is the elastic scattering limit where no extra hadrons are produced by the final state. In the next section we will investigate the small x region, where the contribution of final states that produce extra hadrons will be important.

4. Small x and multi-hadronic states

The supergravity approximation is valid if $x > (gN)^{-1/2}$ so if the 't Hooft constant gN is large enough, we can still use this approximation for a region where $x \ll 1$. In this regime the center of mass energy squared $s \approx q^2/x$ is very large. So we expect the production of a large number of hadrons.

In our formulation there is only one field Φ_X that represents the product of the scattering process. We conjecture that this state, that has a large number of constituents, will evolve into multi-hadronic states, but we will not describe this evolution. This would require the inclusion of more fields in the interaction action.

For a fixed value of the energy and momentum of the final state, the maximum number of hadrons that can be produced by the final state occur when there is no relative motion and all the extra hadrons have the minimum mass Λ. Taking the approximation that the initial hadron has also a mass Λ, the maximum number of produced hadrons is

$$N_{\text{max}} \approx \frac{\sqrt{s}}{\Lambda} \approx \left(\frac{q^2}{x\Lambda^2} \right)^{1/2}. \tag{4.1}$$

This places the limit:

$$0 \leq \rho \leq (N_{\text{max}} - 1). \tag{4.2}$$

The structure function F_2 will be the sum

$$F_2(x, q^2) = F_2^{(\rho=0)}(x, q^2) + \sum_{\rho=1}^{N_{\text{max}}-1} F_2^\rho(x, q^2). \tag{4.3}$$

In the limit $x \to 0$ we have

$$F_2^\rho(x \to 0, q^2) \approx F_2^{(\rho=0)}(x \to 0, q^2) \frac{1}{x^2} \left[\frac{(\Delta)_{\rho-1}}{(\rho-1)!} \right]^2. \tag{4.4}$$

Then

$$\sum_{\rho=1}^{N_{\text{max}}-1} F_2^\rho(x \to 0, q^2) \approx F_2^{(\rho=0)}(x \to 0, q^2) \frac{1}{x^2} N_{\text{max}}^{(2\Delta-1)}. \tag{4.5}$$

Using these results (and eq. (2.24)) we find

$$F_2(x, q^2) \approx \pi C_0 \mathcal{Q}^2 \left(\frac{q^2}{\Lambda^2} \right)^{1/2} x^{-1/2}. \tag{4.6}$$

The total cross section is related to the structure function F_2 by

$$\sigma(q^2, x) = 4\pi^2 \alpha_{EM} \frac{F_2(x, q^2)}{q^2}. \tag{4.7}$$

Then our result for small x implies that

$$\sigma(q^2, x) \sim (q^2 x^\lambda)^{\gamma_s - 1}, \tag{4.8}$$

with $\lambda = 1$ and $\gamma_s = 1/2$. So we find a scaling law for the total cross section different from the case $\Delta' = \Delta$ of eq. (2.25). This scaling is similar to geometric scaling.

It is interesting to compare our scaling with other scalings obtained in different kinematical regimes (at large gN). In ref. [5], in the regime $\exp(-\sqrt{gN}) \ll x \ll (gN)^{-1/2}$ the DIS structure function is

$$F_2 = \frac{2\Delta + 3}{\Delta + 2} \frac{\pi^2 \zeta |C_i|^2}{2(4\pi gN)^{1/2}} \frac{1}{x} \left(\frac{\Lambda^2}{q^2} \right)^{\Delta-1} \mathcal{I}_{1,2\Delta+3} \tag{4.9}$$

where ζ is a dimensionless constant and

$$\mathcal{I}_{j,n} \equiv \int_0^\infty d\omega \, \omega^n K_j^2(\omega) = 2^{n-2} \frac{\Gamma(\frac{n+1}{2} + j)\Gamma(\frac{n+1}{2} - j)\Gamma^2(\frac{n+1}{2})}{\Gamma(n+1)} \,. \tag{4.10}$$

So the total cross section is proportional to

$$\frac{F_2(x,q^2)}{q^2} \sim (q^2 x^\lambda)^{\gamma_s - 1} \,. \tag{4.11}$$

This corresponds to a scaling with $\lambda = 1/\Delta$ and $\gamma_s = 1 - \Delta$. For a baryon in QCD $\Delta = 3$. If this above result could be extended to this case one would find $\lambda = 1/3$ and $\gamma_s = -2$.

A recent article [17] considered the regime $x < \exp(-\sqrt{gN})$ with large gN and N large but finite. They studied the problems of saturation and unitarity in the scattering amplitudes, problems that are related to geometric scaling. In particular they found, in the vicinity of the saturation line, a scaling law with $\lambda = 1$ and $\gamma_s = -1$.

Acknowledgments

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